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Topology and its Applications 95 (1999) 155–163

TOPOLOGY AND ITS APPLICATIONS

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Singularities cancellation on wave fronts

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Received 25 July 1997; received in revised form 17 October 1997

Abstract

We show that any Legendre knot in the contact manifold of cooriented contact elements of a surface M is, *up to stabilization*, Legendre-isotopic to a Legendre knot whose projection on M (wave front) is an immersion, provided that it is Legendre-homotopic to such a knot. As a consequence, we obtain that each ambient isotopy class of knots contains Legendre representatives with immersed wave fronts. We also show that similar results do not hold in the context of the manifold of *noncooriented* contact elements. © 1999 Elsevier Science B.V. All rights reserved.

Keywords: Wave fronts; Legendrian knots; Contact topology

AMS classification: Primary 53C15, Secondary 57

1. Introduction and statement of the results

A cooriented contact element at some point q of a surface M is a cooriented line in T_qM. The space ST^*M of all cooriented contact elements of M is the fibrewise spherization of the cotangent bundle T^*M. It has a natural contact structure (i.e., a nowhere integrable plane field), defined by the following construction. Denote by $\pi : ST^*M \to M$ the natural projection. A vector X, tangent to ST^*M at some contact element belongs to the contact distribution if $\pi_*X \in TM$ lies in the contact element. An embedding of the circle which is everywhere tangent to this contact structure is called a *Legendre knot*. A closed, connected, immersed and cooriented curve on M lifts generically to a Legendre knot in ST^*M. The lifted knot is the set of all cooriented contact elements which are tangent to the curve (with a consistent coorientation). The projection of a Legendre knot to M is called a wave front (or simply a *front*). In general it is not an immersion. The generic singularities are semi-cubic cusps and transverse self-intersections.

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PII: S0166-8641(97)00276-9

Theorem A. *Consider a Legendre knot l in ST^*M. The following three assertions are equivalent*:

(i) *The Maslov index $\mu(l)$ vanishes.*

(ii) *l is* Legendre-homotopic *to the Legendre lift of some cooriented immersed curve on M.*

(iii) *l is*, up to stabilization, Legendre-isotopic *to the Legendre lift of some cooriented immersed curve on M.*

The precise definition of the terminology is given in Section 2.

Theorem B. *Each knot in ST^*M is ambient isotopic to the Legendre lift of some cooriented immersed curve on M.*

When $M = \mathbb{R}^2$, Theorem B was proved previously by Shumakovich [10] and Chmutov, Goryunov and Murakami [4]. It emphasizes a relationship between the theory of curves on a surface on M and knot theory in ST^*M, in the spirit of Arnold's paper [2].

Consider the "long" cusped front depicted in Fig. 1 (left). This example is due to F. Aicardi. The author tried unsuccessfully to remove the cusps of this front without breaking the Legendre knot type. Should it be impossible, the following remark would be noticeable:

Fig. 1. An example by F. Aicardi.

Remark C. The two long Legendre knots whose fronts are depicted in Fig. 1 cannot be distinguished by Vassiliev invariants of long Legendre knots.

See Section 2 below for the definition of a *long* knot.

Let us now consider noncooriented fronts on M, or, equivalently, Legendre knots in PT^*M, the projectivized cotangent bundle of M. Recall that the contact structure of ST^*M gives rise to a contact structure on its fibrewise quotient PT^*M.

The following result shows that, in contrast with the cooriented context, some Legendre knots in $PT^*\mathbb{R}^2$ are not topologically equivalent to a Legendre knot with an immersed front. A (topological) knot in PT^*M is called *small* if it can be embedded in some embedded ball of PT^*M.

Theorem D. *A Legendre knot in $PT^*\mathbb{R}^2$ whose front is immersed is not small.*

The "lips" front of Fig. 2 (left) is the projection of a small Legendre knot in $PT^*\mathbb{R}^2$. By a Legendre homotopy, it can be deformed into the smooth "figure eight" of Fig. 2 (right).

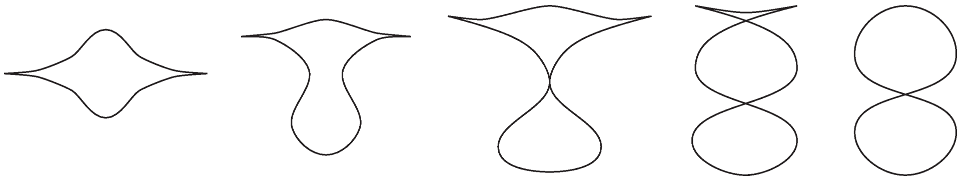

Fig. 2. Fives steps from a path between the “lips” and the “figure eight”.

Fig. 3. The front of l_o and the front of Z.

By Theorem D, such a deformation cannot be the projection of a Legendre isotopy. For instance, the Legendre knot type is broken by the self-tangency that occurs in the third step of the deformation suggested by Fig. 2. However, if one coorients the fronts of Fig. 2, this deformation lifts to a Legendre isotopy in $ST^*\mathbb{R}^2$, since, at the self-tangency point, the coorienting covectors are antipodal.

A crucial ingredient of this note is the idea of stabilization, borrowed from a recent paper by Fuchs and Tabachnikov [7], where it was successfully used for the study of Vassiliev invariants of Legendre knots in the standard contact space $\mathbb{R}^3$. The author acknowledges the strong influence of this paper, and is very grateful to F. Aicardi and to V.I. Arnold who first formulated the problems treated here. The author also thanks the referee for many pertinent remarks.

2. Definitions and preliminary results

We first explain the terminology used in the above statements. For basics of contact geometry, the reader is referred to [1]. Two Legendre knots are *Legendre-homotopic* if there exists a one parameter family of Legendre *immersions* joining them. Two Legendre knots are *Legendre-isotopic* if there exists a one parameter family of Legendre knots joining them, i.e., if they have the same *Legendre knot type*. By a contact isotopy, we mean a one parameter family of contactomorphisms.

We now explain what we mean by *stabilization*. A *long Legendre knot* in $ST^*\mathbb{R}^2$ is a Legendre embedding of the real line, which is standard at infinity, i.e., its front coincides with a fixed oriented and cooriented straight line outside a compact set. The sum $l = l_1 \sharp l_2$ of two long Legendre knots $l_i,\ i \in \{1,2\}$, is well defined.

Denote by l_o the long Legendre knot whose (immersed) front is depicted in Fig. 3 (left).

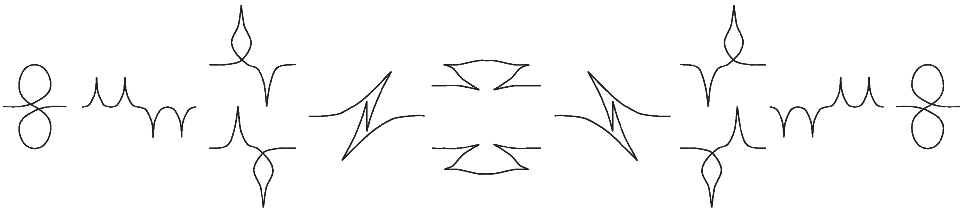

Fig. 4. Useful alter-egos of l_o.

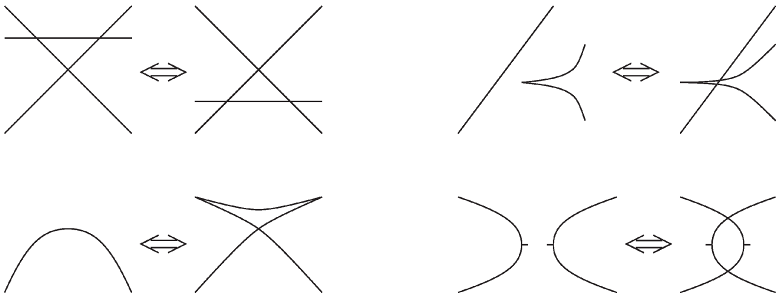

Fig. 5. Legendrian Reidemeister moves.

Definition. We say that a Legendre knot in $ST^*\mathbb{R}^2$ (respectively $PT^*\mathbb{R}^2$) is *flat* with respect to some arbitrary "vertical" direction of $\mathbb{R}^2$ if it is Legendre-isotopic to a Legendre knot whose front is "flat", i.e., has no vertical tangents.

The Z knot of Fig. 3 (right) is an example of a flat long Legendre knot. The knot l_o is also flat. This follows from Fig. 4, which features 12 fronts corresponding to some long Legendre knots which are all Legendre-isotopic to l_o. One can check this by means of the Legendrian version of the Reidemeister moves, which are described in terms of local reconstruction of fronts in Fig. 5 (see [1] for details). Observe that, for the fourth move, the coorientation of the front must be taken into account. When necessary, the coorientation is represented on the figures of this paper by a small "hair", transversal to the front.

Remark. The standard contact space (also known as the space of one-jets of functions on $\mathbb{R}$) is by definition $\mathbb{R}^3 = J^1(\mathbb{R},\mathbb{R}) = T^*\mathbb{R} \times \mathbb{R}$ endowed with the contact form $\alpha = \mathrm{d}u - p\,\mathrm{d}q$, where (u, p, q) are coordinates on $\mathbb{R}^3$. Denote by π the projection $(q, p, u) \to (q, u)$. The projection $\pi(l)$ of a Legendre knot $l \subset \mathbb{R}^3$ is a flat front in the plane spanned by $\partial/\partial q$ and the "vertical" direction $\partial/\partial u$. Conversely, a flat front in the plane lifts to a Legendre knot in $J^1(\mathbb{R},\mathbb{R})$.

The following lemma is a slight generalization of Lemma 4.1 of [7].

Lemma. *A flat long front can "travel" along any other front.*

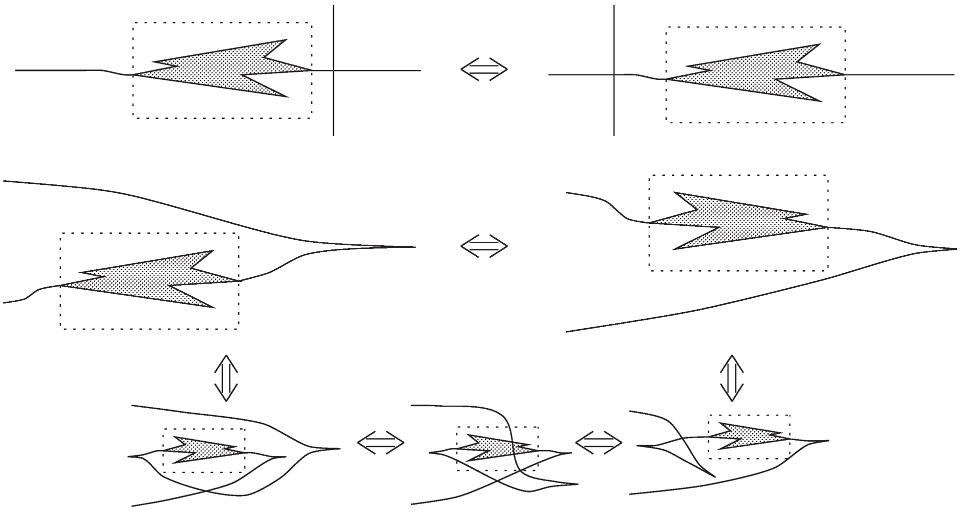

Fig. 6. A flat front (in a box) is free to travel.

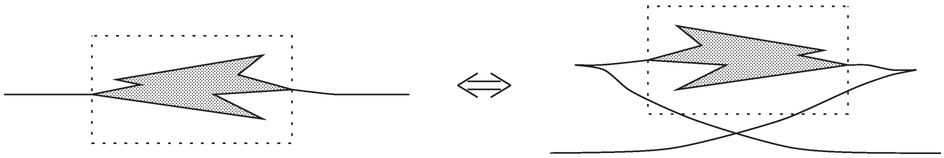

Fig. 7. The "rotation" of a flat front.

Its precise meaning is the following: Put the flat front in a small box. Two fronts on a surface which differs by the elementary moves of Fig. 6 corresponds to Legendre-isotopic knots. In particular, a flat long Legendre knot commutes with any other long Legendre knot.

Proof. The first move is clear, since we can assume that the piece of front which lies inside the box is transverse to some vertical direction. The possibility of the second move follows from the fact that one can "rotate" the box as in Fig. 7. To check this, using the above remark, it is enough to show that the "rotation" of Fig. 7 is realizable by a contact isotopy of $J^1(\mathbb{R},\mathbb{R})$. Suppose that our flat front is standard for $|q| > 1$. Consider the Hamilton function $H(q,p) = (q^2+p^2)/2 \cdot \rho(q^2+p^2)$, where $\rho : (0,+\infty) \to \mathbb{R}$ is a smooth monotonic function such that $\rho(x) = 1$ if $|x| < 2$, $\rho(x) = 0$ if $|x| > 3$. The associated Hamiltonian flow of the symplectic plane $T^*\mathbb{R}$ coincides with a rotation inside the disc $q^2+p^2 < 2$. The contact lift of this flow at time π realizes the desired rotation. □

Since l_o is flat, the following construction is well defined:

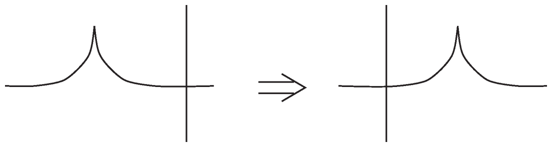

Fig. 8. A cusp can go through a double point, up to homotopy.

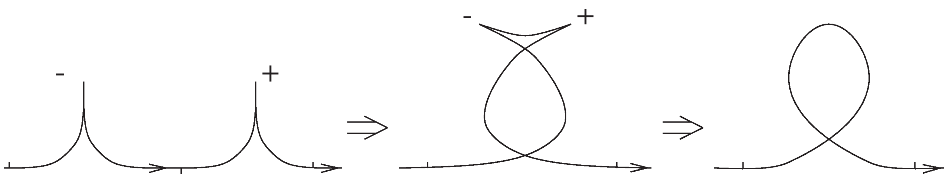

Fig. 9. Cancellation of opposite cusps.

Definition. A *stabilization* of a Legendre knot l in ST^*M is the Legendre knot obtained by cutting the front of l at some nonsingular point and inserting there some small copies of the front of l_o.

We characterize now the Legendre knots which are Legendre-homotopic to a lift of a (cooriented) immersed curve. A (semi-cubic) cusp is called *positive* (respectively *negative*) if, at the cusp point, the front goes through the contact element in the direction of its coorientation (respectively in the opposite direction). As an example, positive and negative cusps are labelled on Fig. 9. The *Maslov index* $\mu(l)$ of a generic Legendre knot l is the algebraic number of cusps.

Proposition. *A Legendre knot l in ST^*M is Legendre-homotopic to the lift of a cooriented immersed curve on a surface M iff $\mu(l) = 0$.*

Proof. During a Legendre homotopy, cusps appear in pairs, with opposite signs. Hence $\mu(l)$ is an invariant of Legendre homotopy and this shows that a Legendre knot l which is Legendre-homotopic to the lift of some cooriented immersed curve on M verifies $\mu(l) = 0$. Conversely, all the cusps of the front of some Legendre knot l can be gathered on a small nonintersecting piece of front, by means of the elementary Legendre homotopies of Fig. 8 (this breaks the Legendre knot type in general). The condition $\mu(l) = 0$ ensures that, on this piece of front, there exists at least one pair of *consecutive* cusps with opposite signs. Trade these pairs for loops as in Fig. 9. After all pairs that can be cancelled have been replaced by loops, gather all the remaining cusps again. And eliminate one or more pair again, etc. Repeat these operations (at each step, the total number of cusps decreases) until the front is immersed. $\square$

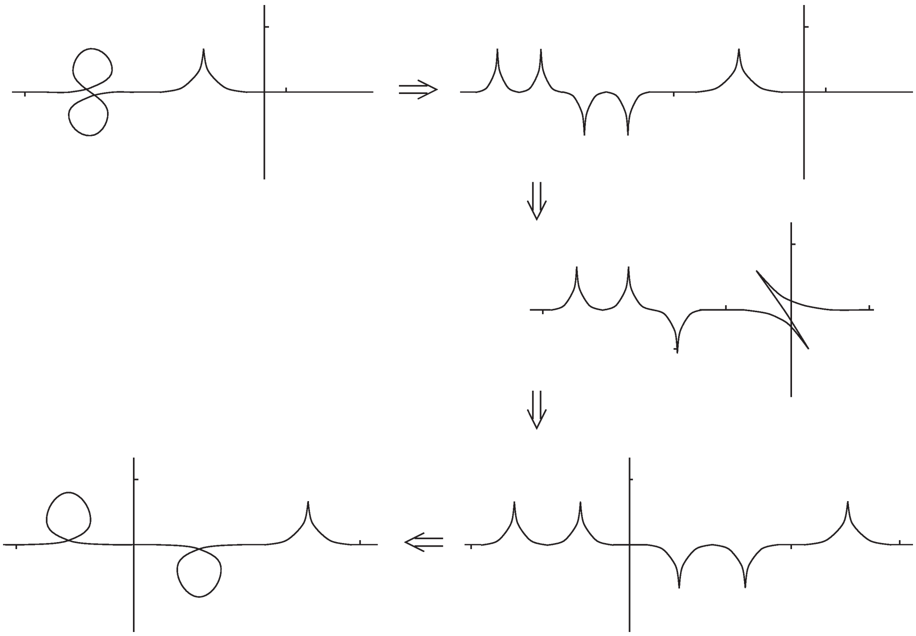

Fig. 10. A cusp can go through a double point, up to stabilization.

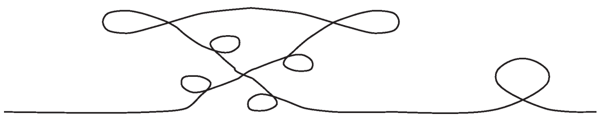

Fig. 11. Aicardi's example, up to stabilization.

3. Proofs

Proof of Theorem A. In view of the proof of the above proposition, it is enough to show that cusps can be gathered up to stabilization. In other words, it is enough to show that a cusp can move through a double point of the front. Fig. 10 shows how to achieve this goal. □

Proof of Theorem B. Recall that any knot k can be $\mathcal{C}^0$-approximated by a Legendre knot l having the same topological type (see [1]). Observe that, since the front of l is coorientable, the total number of cusps is even. Thus the Maslov index $\mu(l)$ is even as well. Hence, adding as many Z as needed on the front, we can suppose that $\mu(l) = 0$. Theorem B then follows from Theorem A, and from the fact that the stabilization procedure does not change the knot type. □

Proof of Remark C. Denote by l_1 (respectively l_2) the Legendre knot whose front is the cusped front of Fig. 1 (left) (respectively the immersed front of Fig. 1 (right)). Apply the stabilization algorithm to l_1. One obtains the immersed front depicted in Fig. 11.

But we recognize that this front is nothing but a stabilization of l_2. Precisely, we have

$$l_o \sharp l_o \sharp l_1 = l_o \sharp l_o \sharp l_2.$$

Since l_o is equivalent to some "zig-zag" as shown in Fig. 4, we recognize the situation when Theorem 4.5 of [7] (suitably generalized to the case $ST^*\mathbb{R}^2$) can be applied: these two Legendre knots cannot be distinguished by Vassiliev invariants of Legendre knots. $\square$

The contact structure of ST^*M is coorientable, and hence, each Legendre knot inherits a natural framing. By [9], the $\mathbb{C}$-valued Vassiliev invariants of Legendre knots in $ST^*\mathbb{R}^2$ are nothing but the restriction to Legendre knots of $\mathbb{C}$-valued Vassiliev invariants of framed knots. But l_1 and l_2 have the same framed knot type. This gives another proof of Remark C in the case of $\mathbb{C}$-valued Vassiliev invariants.

Proof of Theorem D. This section is based on a deep result by Eliashberg and also makes use of a fundamental lemma by Giroux and a result of [6].

(1) *A small Legendre knot can be embedded in a Darboux ball*. Consider a small Legendre knot l. Recall that this means that l is embedded in some embedded ball $B \subset PT^*\mathbb{R}^2$. A *Darboux ball* will denote the image of a contact embedding in $PT^*\mathbb{R}^2$ of the unitary ball of the standard contact space $\mathbb{R}^3$.

Let $S = \partial B$. The characteristic foliation $\mathcal{F}$ of S is by definition the line field with singularities drawn on S by the contact structure. One can assume that this foliation is generic. Using the results of Giroux [8], there exists a ball B', $\mathcal{C}^0$-close to B such that the characteristic foliation of B' is the same as the characteristic foliation of the unitary ball B_o in the standard contact space $\mathbb{R}^3$. By a standard neighborhood argument, the contact structures near the boundaries of B_o and B' are the same up to contactomorphism. Using Eliashberg's classification of tight contact structures in a ball relatively to the boundary [3], we see that B' and B_o are contact-isotopic and that B' is a Darboux ball. This proves (1).

(2) *A Legendre knot which is embedded in a Darboux ball is flat*. Recall that saying that l is *flat* means that l is Legendre-isotopic to some Legendre knot l' such that the front of l' is everywhere transverse to some arbitrary vertical direction of $\mathbb{R}^2$.

Similarly to the remark of Section 2, observe that the set of noncooriented contact elements of the plane which are transversal to some given vertical direction is contactomorphic to the standard contact space $\mathbb{R}^3$. This way of seeing $J^1(\mathbb{R},\mathbb{R})$ as an open subset of $PT^*\mathbb{R}^2$ allows us to associate canonically a Darboux ball in $PT^*\mathbb{R}^2$ to any given vertical direction.

Choose a vertical direction and denote by B_v the associated Darboux ball as above. Consider a Legendre knot l embedded in an arbitrary Darboux ball B'. Denote by P the point in $PT^*\mathbb{R}^2$ which coincides with the center of the ball B' (such a center is well defined when B' is seen as the unitary ball of $\mathbb{R}^3$). Consider any contact isotopy which brings P inside B_v. For ε small enough, the ball B_ε of radius ε around P is engulfed in B_v by this isotopy. But there exists a "radial" contact isotopy of $\mathbb{R}^3$ which engulfs the unitary ball B' inside B_ε. Using the coordinates defined in the remark of Section 2, this isotopy has the form $\Phi_t(u,p,q) = (t^2 \cdot u,\ t \cdot p,\ t \cdot q)$. In other words, there exists a Legendre isotopy in $PT^*\mathbb{R}^2$ which brings l inside B_ε, and hence, in B_v. This proves (2).

(3) *A flat Legendre knot in $PT^*\mathbb{R}^2$ is never Legendre isotopic to the Legendre lift of an immersed manifold*. By Theorem 9 of [6] (see also [5]), the front of a Legendre knot which is contact-isotopic to the lift of an immersed manifold N is never transversal to the levels of a function without critical points. Applying this result with a function whose levels are the vertical lines, we see that such a knot cannot be flat. This proves (3) and Theorem D. □

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